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Exact scaling properties of a hierarchical network model

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We report on the exact results for the degree K, the diameter D, the clustering coefficient C, and the betweenness centrality B of a hierarchical network model with a replication factor M. Such quantities are calculated exactly with the help of recursion relations. Using the results, we show that (i) the degree distribution follows a power law $P_K \sim K^{-\gamma}$ with $\gamma = 1 + \ln M/\ln(M-1)$, (ii) the diameter grows logarithmically as $D \sim \ln N$ with the number of nodes N, (iii) the clustering coefficient of each node is inversely proportional to its degree, $C \propto 1/K$, and the average clustering coefficient is nonzero in the infinite N limit, and (iv) the betweenness centrality distribution follows a power law $P_B \sim B^{-2}$. We discuss a classification scheme of scale-free networks into the universality class with the clustering property and the betweenness centrality distribution.

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A network structure of complex systems has been attracting much research interest [1]. Since the works in Refs. [2,3], it has been recognized that complex systems have neither a regular nor a random network structure. Instead, complex networks found in various areas have a scale-free (SF) structure characterized with a power-law distribution of the degree, see Ref. [1], and references therein. A possible mechanism for the emerging SF structure is suggested by the Barabási-Albert (BA) model [4]. However, the BA network lacks the clustering property that is observed in many real networks. To fill this gap, a hierarchical (H) network model was introduced as a model for SF networks with a clustering property [5]. It is observed that the H network displays a scaling law $C \sim K^{-1}$ between the clustering coefficient C of a node and its degree K and that the average value of C does not vanish in the infinite size limit. The clustering property is also observed in some metabolic networks [6], which is regarded as an evidence for a hierarchical structure [5,7].

On the other hand, the clustering property does not necessarily imply a hierarchical structure. Some SF network models [8–11] have the clustering property, but it is not apparent whether they have a hierarchical structure. Therefore, it is desirable to study other scaling properties of the H network model to establish the universality class for the hierarchical network. This is the purpose of the current work. We derive analytically exact scaling laws for the degree distribution, the diameter D, the clustering coefficient (CC), and the betweenness centrality (BC) B [5]. The CC is a measure of local connectivity or modularity, while the BC reflects a global property [12,13]. The scaling property of both quantities characterizes the universality class for the H network.

The H network is characterized by G (the number of generations) and M (a replication factor) [5]. The network of the Gth generation, denoted as $\mathcal{N}_G$, has $N = M^G$ nodes, which will be labeled by the coordinate G tuple of integers $[\boldsymbol{x}_G] \equiv [x_G \cdots x_1]$ with $0 \leqslant x_i < M$. The network is defined recursively. The first generation consists of one central node $[0]$, which will be referred to as a hub, and $(M-1)$ peripheral nodes $[y]$ with $1 \leqslant y < M$. All nodes are fully connected with each other. Suppose one has $\mathcal{N}_{G-1}$, each node of which is assigned to a coordinate $[\boldsymbol{x}_{G-1}]$. In the next generation, $(M-1)$ copies are added to the network and all their peripheral nodes are connected to the hub of the original unit. Nodes in the original unit are assigned to $[0\boldsymbol{x}_{G-1}]$ and the copies to $[y\boldsymbol{x}_{G-1}]$ with $y = 1, \ldots, M-1$, respectively. It leads to the network $\mathcal{N}_G$. Nodes $[y_G \cdots y_1]$ with $y_i \neq 0$ for all i are the peripheral nodes and $[\mathbf{0}_G]$ is the hub of $\mathcal{N}_G$.

With the coordinate system, geometrical properties of the network can be studied combinatorially. In general, the node connectivity follows the rule:

Rule (1)

$$[x_G \cdots x_{l+1} y_l \cdots y_1] \leftrightarrow [x_G \cdots x_{l+1} y_l \cdots y_{k+1} \mathbf{0}_k]$$

$$(1 \leqslant l \leqslant G \ \text{ and } \ 1 \leqslant k \leqslant l).$$

Rule (2)

$$[x_G \cdots x_2 y_1] \leftrightarrow [x_G \cdots x_2 y_1'] \ \text{ for } \ y_1 \neq y_1'.$$

Above and hereafter, we use x for a dummy variable ranging from 0 to $(M-1)$ and y for one ranging from 1 to $(M-1)$. Rule (1) comes from the fact that all peripheral nodes of $\mathcal{N}_k$ are connected to the hub $\mathbf{0}_k$ during the replication and Rule (2) from the fact that all peripheral nodes of $\mathcal{N}_1$ are fully interconnected.

For a convenience, we classify nodes into four sets: (a) $\mathbb{P}$ for peripheral nodes $[y_G \cdots y_1]$, (b) $\mathbb{LP}l$ (stands for locally peripheral) $(1 \leqslant l < G)$ for nodes of the form $[x_G \cdots x_{l+2} 0\, y_l \cdots y_1]$, (c) $\mathbb{LH}l$ (local hub) $(1 \leqslant l < G)$ for nodes of the form $[x_G \cdots x_{l+2} y_{l+1} \mathbf{0}_l]$, and (d) $\mathbb{H}$ for the hub $[\mathbf{0}_G]$. The size S of each set is given in Table I.

Degree distribution. Using the connection rules, one can easily enumerate the degree, which is the number of neighbors, of each node. All nodes in the same set have the same degree, which are presented in Table I. The nodes in $\mathbb{P}$ and $\mathbb{H}$ contribute to the degree distribution P_K at isolated points of K. For nodes in $\mathbb{LP}l$, it is given by $P_K = M^{-G} S_l |\Delta l / \Delta K_l|$ with $\Delta l = 1$ and $\Delta K_l \equiv K_{l+1} - K_l = 1$. Using K and S in Table I, we find that $P_K \sim \exp[K \ln(1 - 1/M)]$. On the other hand, nodes in $\mathbb{LH}l$ have $K_l \sim (M-1)^l$ and $S_l \sim M^{-l}$. Hence, the degree distribution follows a power law $P_K \sim K^{-\gamma}$ for $M > 2$ with $\gamma = 1 + \ln M/\ln(M-1)$. The nodes in $\mathbb{LH}l$ have larger degrees than those in $\mathbb{LP}l$. Therefore, the total degree distribution follows the power law with the exponent γ in the tail region. In particular, the hub has the largest degree, which scales as $K_{hub} \sim N^{1/(\gamma-1)}$.

TABLE I. The degree K, the clustering coefficient C, the partial betweenness centrality B^0, and the betweenness centrality B of a node in each set and the number of nodes S in each set.

Set	S	K	C	B^0	B
$\mathbb{P}$	$(M-1)^G$	$G+(M-2)$	$\frac{(M-2)(2G+M-3)}{(G+M-2)(G+M-3)}$	$\left(\frac{M}{M-1}\right)^{G-1}-1$	$\simeq 2M^G\left(\frac{M(M^2-1)}{M^3+1}\right)\left\{\left(\frac{M}{M-1}\right)^{G-1}-1\right\}$
$\mathbb{LP}l$	$(M-1)^l M^{G-(l+1)}$	$l+(M-2)$	$\frac{(M-2)(2l+M-3)}{(l+M-2)(l+M-3)}$	$\left(\frac{M}{M-1}\right)^{l-1}-1$	$\simeq 2M^G\left\{\left(\frac{M}{M-1}\right)^{l-1}-1\right\}$ for $l \ll G$
$\mathbb{LH}l$	$(M-1)M^{G-(l+1)}$	$\sum_{n=1}^{l}(M-1)^n$	$\frac{(M-2)}{\sum_{n=1}^{l}(M-1)^n-1}$	$M^{l-1}-1$	$\simeq 2M^G(M^{l-1}-1)$ for $l \ll G$
$\mathbb{H}$	1	$\sum_{n=1}^{G}(M-1)^n$	$\frac{(M-2)}{\sum_{n=1}^{G}(M-1)^n-1}$	0	$\simeq\left\{\frac{2}{M+1}+\frac{M^2(M-2)}{M^2-1}\ln\frac{M^2}{M^2-M+1}\right\}M^{2G}$

Shortest path and diameter. We first consider the shortest path from an arbitrary node $[x_G\cdots x_1]$ to the hub $[\mathbf{0}_G]$. One can reach the hub by flipping the coordinate using Rules (1) and (2) successively. The shortest path has the minimum number of steps, which is called the distance. Note that a step using Rule (2) always leads to a detour. Hence, it must not be used in finding the shortest path to the hub. Following Rule (1), one may flip the *consecutive* digits (including x_1) that are all zero to nonzero values, or vice versa. The whole consecutive digits $x_i\cdots x_1$ with $x_{1\leqslant j\leqslant i}=0$ ($x_{i\leqslant j\leqslant i}\neq 0$) and $x_{i+1}\neq 0$ ($x_{i+1}=0$) will be referred to as a zero (nonzero) *domain*. Then, one can reach the hub in minimal steps by flipping a zero or nonzero domain to a nonzero or zero domain successively. The domain size increases at each flip until one reaches the hub, a zero domain of size G. Since a zero domain can be flipped to any nonzero domain, the shortest path has a large degeneracy in general.

The process resembles domain coarsening in magnetic systems. To complete this analogy, we map the coordinate $[x_G\cdots x_1]$ onto a spin state of the $(Q=M)$-state Potts model in one-dimensional lattice of size G and assign the energy with the Hamiltonian

$$\mathcal{H}=\sum_{i=1}^{G}\{1-\delta(\delta(x_i,0),\delta(x_{i+1},0))\}, \tag{1}$$

where $\delta(\cdots)$ is the Kronecker delta symbol and $x_{G+1}\equiv 0$ is a fixed ghost spin. Then the distance between a node to the hub is given by the energy of the spin state. Therefore, the mean node-to-hub distance is given by the average energy of the spin system in the infinite temperature limit: $D_H(G)=-[\partial \ln Z(\beta;G)/\partial\beta]_{\beta=0}$ with the partition function $Z(\beta;G)=\Sigma_{x_G}e^{-\beta\mathcal{H}[x_G]}$. It can be calculated using a transfer matrix method. After some algebra, we obtain that

$$D_H(G)=\frac{2(M-1)}{M^2}G+\frac{(M-1)(M-2)}{M^2}. \tag{2}$$

The shortest path between arbitrary nodes $[\boldsymbol{x}_G]=[x\boldsymbol{x}_{G-1}]$ and $[\boldsymbol{x}'_G]=[x'\boldsymbol{x}'_{G-1}]$ can be found recursively. If $x\neq x'$, all paths connecting them pass through the hub $[\mathbf{0}_G]$. So, the shortest path is given by a shortest path from one node to $[\mathbf{0}_G]$ followed by a shortest path from $[\mathbf{0}_G]$ to the other. If $x=x'$, one can restrict the shortest path within a subnetwork of all nodes $[x''\boldsymbol{x}''_{G-1}]$ with $x''=x$ for a path utilizing other nodes with $x''\neq x$ does not reduce a path length [14]. Note that the subnetwork with all links to other nodes in $\mathcal{N}_G$ disabled has that same structure as $\mathcal{N}_{G-1}$. Therefore, the distance satisfies the recursion relation

$$d([\boldsymbol{x}_G],[\boldsymbol{x}'_G])=\delta'(x,x')\{d([\boldsymbol{x}_G],[\mathbf{0}_G])+d([\boldsymbol{x}'_G],[\mathbf{0}_G])\}+\delta(x,x')d([\boldsymbol{x}_{G-1}],[\boldsymbol{x}'_{G-1}]), \tag{3}$$

where $\delta'(\cdots)\equiv 1-\delta(\cdots)$ in shorthand notation. Summing up over all node pairs, we obtain the recursion relation $D(G)=2M^{-1}(M-1)D_H(G)+M^{-1}D(G-1)$ for the diameter (mean node-to-node distance) with the solution

$$D(G)=\frac{4(M-1)G}{M^2}+\frac{2(M-3)}{M}-\frac{(M^2-M-4)}{M^{G+1}}. \tag{4}$$

In the infinite $N=M^G$ limit, we find that $D\simeq 2D_H$ and that the diameter scales logarithmically with N. It is a characteristic of the hierarchical network. For conventional (nonhierarchical) SF networks, the diameter scales sublogarithmically for $\gamma\leqslant 3$ [15,16].

Clustering coefficients. The CC of a node with K neighbors is given by $C=2N_e/K(K-1)$, where N_e is the number of existing edges between K neighbors. Using the connection rules, it is straightforward to calculate the CC of each node. Nodes in the same set have the same value of C, which are presented in Table I.

Using the results in Table I, the degree dependence of the CC is easily obtained. For nodes in $\mathbb{P}$ and $\mathbb{LP}l$, we obtain that $C(K)=(M-2)(2K-M+1)/K(K-1)$ with $M-1\leqslant K\leqslant G+(M-2)$. So, for large $K\gg M$, their CC's are inversely proportional to the degrees, $C(K)\simeq 2(M-2)/K$. The CC's of nodes in $\mathbb{LH}l$ and the hub are exactly given by $C(K)=(M-2)/(K-1)$. The scaling law $C\simeq cK^{-1}$ holds for both cases, but the coefficient c differs by a factor of 2 (cf. Fig. 2(b) in Ref. [5]).

The average CC is given by $\overline{C} \equiv M^{-G}(S_{\mathbb{P}}C_{\mathbb{P}} + \Sigma_l(S_{\mathbb{LP}l}C_{\mathbb{LP}l} + S_{\mathbb{LH}l}C_{\mathbb{LH}l}) + S_{\mathbb{H}}C_{\mathbb{H}})$. In the infinite size limit $(G \to \infty)$, it converges to a *nonzero* value

$$\overline{C} = \left(1 - \frac{2}{M}\right) \sum_{l=1}^{\infty} \frac{(2l+M-3)(1-1/M)^l}{(l+M-2)(l+M-3)} + \left(1 - \frac{1}{M}\right) \sum_{l=1}^{\infty} \frac{(M-2)^2}{M^l((M-1)^{l+1} - 2M + 3)}. \quad (5)$$

Numerically $\overline{C} = 0.719\,282\ldots$ and $0.741\,840\ldots$ for $M = 4$ and 5, respectively. $\overline{C}$ converges to 1 as $M \to \infty$.

Betweenness centrality. The BC of a node is the sum of weights of the shortest paths between all node pairs that pass through the node. For a given node pair, all the degenerate shortest paths connecting them are weighted with the inverse of the degeneracy. First of all, we calculate a so-called partial BC B^0, which is obtained from a partial sum over all the shortest paths between the hub and the others. It is calculated easily, since all the shortest paths can be constructed using the domain-coarsening picture.

Again, each node in the same set has the same value of B^0. (a) A node $u = [y_G \cdots y_1]$ in $\mathbb{P}$ may belong to a shortest path to the hub from nodes $[y_G \cdots y_{l+2} 0 x'_l \cdots x'_1]$ with $0 \leqslant l \leqslant G-2$ and arbitrary x'_i. The domain-coarsening process leads them to $[y_G \cdots y_{l+2} \mathbf{0}_{l+1}]$ at an intermediate step. Then, the zero domain $\mathbf{0}_{l+1}$ flips to a nonzero domain in the next step with probability $(M-1)^{-(l+1)}$ passing through the node u. Hence, each node contributes $(M-1)^{-(l+1)}$ to B^0_u. Summing up all contributions, we obtain the result in Table I. (b) A node $[x_G \cdots x_{l+2} 0 y_l \cdots y_1]$ in $\mathbb{LP}l$ may belong to the shortest path from nodes $[x_G \cdots x_{l+2} 0 y_l \cdots y_{m+2} 0 x'_m \cdots x'_1]$ with $0 \leqslant m \leqslant l-2$ and arbitrary x'_i to the hub. Following the same idea as in (a), one can easily obtain B^0, see Table I. (c) A node $u = [x_G \cdots x_{l+2} y_{l+1} \mathbf{0}_l]$ in $\mathbb{LH}l$ belongs to "all" the shortest paths from nodes $[x_G \cdots x_{l+2} y_{l+1} 0 x'_{l-1} \cdots x'_1]$ with arbitrary x'_i, except for u itself. So, $B^0_u = M^{l-1} - 1$. (d) Trivially, $B^0 = 0$ for the hub.

By eliminating the parameter l in Table I, we obtain that $B^0(K) = [M/(M-1)]^{K-M+1}$ for nodes in set $\mathbb{LP}l$. It is diverging exponentially with K. On the other hand, P_K for nodes in $\mathbb{LP}l$ decays exponentially. So, P_{B^0} decays algebraically. Explicitly, it is obtained from $P_{B^0} = M^{-G} S_l |\Delta l / \Delta B^0_l|$, which yields that

$$P_{B^0} = \left(\frac{M-1}{M}\right)^2 \frac{1}{(B^0+1)^2}. \quad (6)$$

We obtain the relation $B^0(K) = -1 + M^{-1}[1 + K(M-2)(M-1)^{-1}]^{\gamma-1}$ for nodes in the set $\mathbb{LH}l$. So the distribution is given by

$$P_{B^0} = \frac{1}{M^2} \frac{1}{(B^0+1)^2}. \quad (7)$$

Therefore, the partial betweenness centrality has the power-law distribution with the exponent 2.

The mean node-to-node distance was obtained using the mean node-to-hub distance. We apply a similar idea for the BC. Introduce a character function $\chi_G([\boldsymbol{x}'_G],[\boldsymbol{x}''_G];[\boldsymbol{x}_G])$ to denote the fraction of paths passing through $[\boldsymbol{x}_G]$ among all the shortest paths between $[\boldsymbol{x}'_G]$ and $[\boldsymbol{x}''_G]$ in $\mathcal{N}_G$. The $[\boldsymbol{x}_G]$ dependence will be assumed implicitly. Then, the BC of a node $[\boldsymbol{x}_G] = [x \boldsymbol{x}_{G-1}]$ can be written formally as $B_{[\boldsymbol{x}_G]} = \Sigma_{\boldsymbol{x}'_G, \boldsymbol{x}''_G} \chi_G([\boldsymbol{x}'_G],[\boldsymbol{x}'_G])$. The hub will be considered separately later, and we assume that $[\boldsymbol{x}_G]$ is not the hub for the time being. Decompose the sum $\Sigma_{\boldsymbol{x}'_G}$ into $\Sigma_{x'} \Sigma_{\boldsymbol{x}'_{G-1}}$ and similarly for $\boldsymbol{x}''_G$. When $x' \neq x''$, all the shortest paths between $[x' \boldsymbol{x}'_{G-1}]$ and $[x'' \boldsymbol{x}''_{G-1}]$ pass through the hub, which yields that $\chi_G([x' \boldsymbol{x}'_{G-1}],[x'' \boldsymbol{x}''_{G-1}]) = \chi_G([x' \boldsymbol{x}'_{G-1}],[\mathbf{0}_G]) + \chi_G([x'' \boldsymbol{x}''_{G-1}],[\mathbf{0}_G])$. When $x' = x''$ and $[\boldsymbol{x}_G]$ is not the hub, the summands are nonzero only when $x' = x'' = x$. Using these properties, we obtain that

$$B_{[\boldsymbol{x}_G]} = 2(M-1) M^{G-1} B^0_{[\boldsymbol{x}_G]} + \sum_{\boldsymbol{x}'_{G-1}, \boldsymbol{x}''_{G-1}} \chi_G([x \boldsymbol{x}'_{G-1}],[x \boldsymbol{x}''_{G-1}]). \quad (8)$$

One might be tempted to identify the second term as $B_{[\boldsymbol{x}_{G-1}]}$, i.e., the BC defined on $\mathcal{N}_{G-1}$. However, this is not correct for $x \neq 0$, since some pairs of $[\boldsymbol{x}'_{G-1}]$ and $[\boldsymbol{x}''_{G-1}]$ may have degenerate shortest paths passing through the hub $[\mathbf{0}_G]$ that is not present at $\mathcal{N}_{G-1}$. This was explained when we discussed the diameter. For example, $[y1111]$ and $[y1222]$ with nonzero y are connected via $[y1000]$ and $[y0000]$, and also via $[00000]$, so each path has the weight 1/3. However, if one ignores the path via $[00000]$, the other paths would have weight 1/2, and hence the nodes $[y1000]$ and $[y0000]$ would have larger value of the BC.

Therefore, the second term in Eq. (8) should be written as a sum of $B_{[\boldsymbol{x}_{G-1}]}$ and a quantity that compensates for the change in the degeneracy of the shortest paths. As the example shown, the compensation is necessary only for nodes of the form $[\boldsymbol{x}_G] = [y_G \cdots y_{l+1} \mathbf{0}_l]$ with $1 \leqslant l < G$. For such nodes, after enumerating all degeneracy carefully, Eq. (8) becomes

$$B_{[\boldsymbol{x}_G]} = 2(M-1) M^{G-1} B^0_{[\boldsymbol{x}_G]} + B_{[\boldsymbol{x}_{G-1}]} - \sum_{k=1}^{l-1} \frac{(M-2)(M-1)^k M^{2(l-k)}}{(G-l+k)(G-l+k-1)}. \quad (9)$$

The hub gains from such degenerate shortest paths, which lead to a similar recursion relation

$$B_{[\mathbf{0}_G]} = (M-1)(M^{2G-1} - 2M^{G-1}) + B_{[\mathbf{0}_{G-1}]} + (M-2) M^{2G} \sum_{k=2}^{G-1} \frac{(M-1)^k}{k M^{2k}}. \quad (10)$$

For other nodes, we have the simple relation

$$B_{[\boldsymbol{x}_G]} = 2(M-1) M^{G-1} B^0_{[\boldsymbol{x}_G]} + B_{[\boldsymbol{x}_{G-1}]}. \quad (11)$$

With the recursion relations, we now readily calculate the BC of all the nodes exactly. Since the exact expressions are lengthy, we present only the leading-order contribution in the large G limit in Table I. Note that $B \simeq 2M^G B^0$ for most nodes in $\mathbb{LP}l$ and $\mathbb{LH}l$ in the large G limit. The proportionality relation implies that the BC follows the same power-law distribution as B^0:

$$P_B \sim B^{-2}. \tag{12}$$

Summary and discussions. We have studied the exact scaling properties of the H network introduced in Ref. [5]. We have shown that the H network has the clustering property: The average value of the CC is nonzero in the infinite network size limit and the CC exhibits the scaling law $C \sim K^{-\beta}$ with $\beta = 1$. We have also shown that the BC follows the power-law distribution $P_B \sim B^{-\eta}$ with $\eta = 2$. Both scaling properties characterize the H network model.

The BC proved to be useful in classifying SF networks into the universality class. The BC distribution exponent is universal and has the value either $\eta \simeq 2.2$ in the class I or $\eta \simeq 2.0$ in the class II [13]. Combining the scaling properties of the CC and the BC, we suggest that there exist four classes, that is, I-C, I-NC, II-C, and II-NC (C for clustered and NC for nonclustered networks) [17]. The H network model then belongs to the class II-C. The Internet at the autonomous system level and some metabolic networks of archaea display both scaling behaviors with $\beta \simeq 1.0$ [5] and $\eta \simeq 2.0$ [13]. So they belong to the same class II-C, which is a stronger evidence for a hierarchical structure [5]. The BA network with $m = 1$ [4] and the deterministic tree network [18] have $C = 0$ and $\eta = 2$ [19], thus they are members of the class II-NC. The BA network with $m \geqslant 2$ has vanishing CC [9,10] and $\eta \simeq 2.2$ [13], and belongs to the class I-NC.

Literatures suggest that the metabolic networks of bacteria and eukaryotes [13,7] and the co-authorship network in the field of neuroscience [13,20] might belong to the class I-C. With the existence of the class I-C, the clustering property does not necessarily imply the hierarchical structure. Therefore, it is important to establish the class I-C firmly. Further studies on the BC distribution in model networks with the clustering property, such as the Holme-Kim model [8] and Klemm-Eguíluz model [9], are required. Further studies are also necessary to reveal the similarity or dissimilarity between the metabolic networks of archaea and those of bacteria and eukaryotes, which have different BC distributions.

Goh *et al.* [13] suggested that the topology of the shortest pathways be a universal characteristic of SF networks. They found a chainlike structure for networks in the class II ($\eta = 2$). It was presumed that the chainlike structure leads to a linear mass-distance relation $m(d) \simeq Ad$, where $m(d)$ is the mean number of nodes along all the shortest paths between a node pair separated by a distance d. We also found that the shortest pathways of the H network have a chainlike structure. For example, a set of all shortest paths from $[001010]$ to the hub in $\mathcal{N}_6$ is given by $[001010] \rightarrow_1 [00101y] \rightarrow_2 [001\mathbf{0}_3] \rightarrow_3 [001yyy] \rightarrow_4 [\mathbf{0}_6]$ with arbitrary nonzero y's. It has a chainlike structure, but steps 1 and 3 introduce blobs whose size increases *exponentially* as one proceeds. We could show that $m(d)$, averaged over all nodes separated by the distance d from the hub, satisfies an inequality $m(d) \geqslant a(M-1)^d$ with a positive constant a [21]. So a topological characterization other than the mass-distance relation is necessary to characterize the chainlike structure observed in the class II-C.

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